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Accurate and rapid thermal regenerator calculations

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Abstract—Means are described whereby the speed of calculation of the performance of thermal regenerators can be considerably increased using the method of Hill and Willmott (*Int. J. Heat Mass Transfer* **30**, 241–249 (1987)) which was developed from the method of Razelos (*Wärme- und Stoffübertr.* **12**, 59–71 (1979)). In particular the number of simultaneous linear equations which must be solved for the non-symmetric case is halved. Advantage is taken of Richardson's extrapolation method.

1. OVERVIEW

REFERENCE [1] presented recently a stable and accurate closed method of solution to the partial differential equations developed by Hausen [2]:

$$\frac{\partial T}{\partial \eta} = t - T \quad \text{where } \eta = \frac{hA}{MC}\left(\tau - \frac{my}{WL}\right) \tag{1}$$

$$\frac{\partial t}{\partial \xi} = T - t \quad \text{where } \xi = \frac{hAy}{WSL}. \tag{2}$$

The physical assumptions embodied in this linear model are set out by Hausen [2]. The method of solution followed the approach adopted by Razelos [3] but in which equation (2) was replaced with a finite difference representation employing the trapezoidal rule with $(N+1)$ nodes. It is this trapezoidal method which provides excellent properties of numerical stability (see p. 74 of ref. [4]). The transformation $\Psi_n = \exp(\eta)T_n$ is then employed to derive a set of ordinary differential equations in the time domain.

The solution to this set of ordinary differential equations enables expressions for the solid temperature to be developed in which the only unknowns are $(N+1)$ constants of integration for each period of regenerator operation, making $2(N+1)$ in all for a complete cycle. By applying the counter-flow boundary conditions $2(N+1)$ equations in the $2(N+1)$ unknowns can be derived from which these unknown constants of integration can be determined.

In the case of the symmetric regenerator, $\Lambda = \Lambda'$ and $\Pi = \Pi'$, where Λ and Π are the *reduced length* and the *reduced period* respectively defined by Hausen [2] as

$$\Lambda = \frac{hA}{WS} \tag{3}$$

$$\Pi = \frac{hA}{MC}\left(P - \frac{m}{W}\right) \tag{4}$$

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the solution of only $(N+1)$ equations is required. It has already been shown that computational speed of this 'robust' method of solution is good for symmetric regenerator calculations [1]. When it is extended to the unsymmetric case the method becomes computationally slow, especially if large values of N ($N \geqslant 30$) are required. The method remains robust however.

This paper is concerned with numerical techniques for improving the computational speed of this method of solution. It is shown that, by manipulating the matrix of coefficients, for the cases of both the symmetric and unsymmetric regenerators, the major work load associated with the method of solution can be reduced by 50–80%. Furthermore, by employing Richardson's extrapolation technique (see p. 186 of ref. [4]) to values of the thermal effectiveness, η_{reg}, computed employing only small values of N, very accurate values of η_{reg} can be obtained at minimal computational cost. Also it is shown how the Richardson extrapolation technique can be applied to the solid and fluid temperatures in order to yield a precise estimate of the complete temperature profile of the regenerator at cyclic equilibrium.

2. THE METHOD IN BRIEF

A detailed description of this method of solution of equations (1) and (2) is presented in ref. [1]. It is necessary here, therefore, to provide only an outline of the method and this is set out below. The regenerator is split into N sections, each of equal width $\Delta\xi = \Lambda/N$, and the temperatures of the solid and fluid considered at the $(N+1)$ positions $\xi = n\Delta\xi$ $(0 \leqslant n \leqslant N)$. The solid and fluid temperatures at time η and position $\xi = n\Delta\xi$ are denoted by T_n and t_n, respectively. The trapezoidal rule is employed to replace equation (2) to give

$$t_{n+1} = bt_n + a(T_{n+1} + T_n) \quad (0 \leqslant n \leqslant N-1) \tag{5}$$

NOMENCLATURE

a	trapezoidal discretization constant
A	heat transfer surface area [m^2]
b	trapezoidal discretization constant
C	specific heat of matrix [$J\,kg^{-1}\,K^{-1}$]
h	bulk heat transfer coefficient [$J\,s^{-1}\,m^{-2}\,K^{-1}$]
K_n	constant of integration
L	length of regenerator [m]
m	mass of fluid in regenerator channels [kg]
M	mass of matrix [kg]
N	total number of regenerator segments
P	duration of a period [s]
S	specific heat of fluid [$J\,kg^{-1}\,K^{-1}$]
t	fluid temperature [K]
T	solid temperature [K]
W	mass flow rate of fluid [$kg\,s^{-1}$]
y	distance down the regenerator [m].

Greek symbols

η	dimensionless time
η_{reg}	thermal effectiveness
Λ	reduced length
ξ	dimensionless distance
Π	reduced period
τ	time [s]
Ψ	transformation of T.

Superscript

$'$	cold period.

Subscript

n	nth node.

where

$$a = \frac{\Delta\xi}{2+\Delta\xi}, \quad b = \frac{2-\Delta\xi}{2+\Delta\xi}. \tag{6}$$

The temperature at $n = 0$ is the known fluid inlet temperature denoted by t_{in} (constant). Using the transformation $\Psi_n = \exp(\eta)T_n$ the following set of ordinary differential equations can be derived:

$$\frac{d\Psi_0}{d\eta} = \exp(\eta)t_{in} \tag{7}$$

$$\frac{d\Psi_n}{d\eta} = a\Psi_n + a\Psi_{n-1} + b\frac{d\Psi_{n-1}}{d\eta} \quad (1 \leqslant n \leqslant N). \tag{8}$$

Upon solving this set of ordinary differential equations the following expressions for the solid temperature can be obtained:

$$T_0 = \exp(-\eta)K_0 + t_{in} \tag{9}$$

$$T_n = \exp(-\eta)(-1)^n K_0 + \exp((a-1)\eta)\left\{K_n + \sum_{j=1}^{n-1} K_j\alpha_{n-j}\right\} + t_{in} \tag{10}$$

where α_r is given by

$$\alpha_r = b^r \sum_{k=1}^{r} \binom{r-1}{k-1}\left(\frac{a(1+b)\eta}{b}\right)^k \frac{1}{k!} \quad (1 \leqslant r \leqslant N-1) \tag{11}$$

and K_j $(0 \leqslant j \leqslant N)$ are the unknown constants of integration. The α_r values can be computed employing the algorithm described in Appendix A.

In order to determine the unknown constants of integration K_j, K_j', the counter-flow reversal conditions

$$\left.\begin{aligned} T_n(0) &= T'_{N-n}(\Pi') \\ T'_n(0) &= T_{N-n}(\Pi) \end{aligned}\right\} \tag{12}$$

are applied to solid equations (9) and (10) to give $2(N+1)$ equations in the $2(N+1)$ unknowns K_j, K_j' which can be expressed concisely as

$$\begin{bmatrix} \mathbf{L} & \mathbf{F}' \\ \mathbf{F} & \mathbf{L} \end{bmatrix}\begin{bmatrix} \underline{k} \\ \underline{k}' \end{bmatrix} = \begin{bmatrix} \underline{y} \\ -\underline{y} \end{bmatrix} \tag{13}$$

where

$$y_i = t'_{in} - t_{in} \tag{14}$$

$$\underline{k} = [K_0, K_1, \ldots, K_N]^T$$

$$\underline{k}' = [K'_0, K'_1, \ldots, K'_N]^T. \tag{15}$$

$\mathbf{L}$, $\mathbf{F}$ are $(N+1) \times (N+1)$ matrices given by

$$\mathbf{L}(i,j) = \begin{cases} (-1)^i & \text{if } j = 0 \\ 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \tag{16}$$

$$\mathbf{F}(i,j) = \begin{cases} -\exp(-\Pi)(-1)^{N-i} & \text{if } j = 0 \\ -\exp((a-1)\Pi) & \text{if } (i+j) = N, j > 0 \\ -\exp((a-1)\Pi)\alpha_{N-i-j} & \text{if } (i+j) < N, j > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{17}$$

The α_r values in $\mathbf{F}$ are computed at $\eta = \Pi$. $\mathbf{F}'$ has the same form as $\mathbf{F}$ with Π', a', b', α_r' replacing Π, a, b, α_r. As $(0 < a < 1)$ the elements of matrix $\mathbf{F}$ are all of a similar magnitude.

For the symmetric regenerator it can be shown that

$$\underline{k} = -\underline{k}' \tag{18}$$

and only the top or bottom 'half' of the system of equations (13) need be considered to give

$$(\mathbf{L}-\mathbf{F})\underline{k} = \underline{y} \tag{19}$$

matrices $\mathbf{F}$ and $\mathbf{F}'$ being equal when $\Lambda = \Lambda'$ and $\Pi = \Pi'$.

This is a system of only $(N+1)$ equations in the $(N+1)$ unknowns K_j $(0 \leqslant j \leqslant N)$. The major work load associated with solving N such linear algebraic equations is proportional to $N^3/3$ (see p. 55 of ref. [5]) and thus eight times the CPU time is required to solve the unsymmetric case $\{(2N)^3/3\}$ as compared with the symmetric case $\{N^3/3\}$; this becomes an acute issue when N is large.

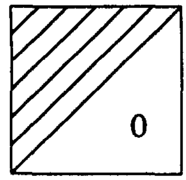

FIG. 1. Form of triangular matrices $\mathbf{F}$ and $\mathbf{F}'$.

3. COMPUTING THE THERMAL EFFECTIVENESS, η_{reg}

Using the initial solid temperatures in the manner proposed by Iliffe [6], η_{reg} is evaluated

$$\eta'_{\text{reg}} = \frac{\Lambda'}{\Pi'}\left[\frac{1}{\Lambda'}\int_0^{\Lambda'} T'(0,\xi)\,\mathrm{d}\xi - \frac{1}{\Lambda}\int_0^{\Lambda} T(0,\xi)\,\mathrm{d}\xi\right]. \tag{20}$$

Employing the trapezoidal rule this becomes

$$\eta'_{\text{reg}} = \frac{\Lambda'}{\Pi'}\frac{1}{N}\left\{\left(\frac{T'_0}{2} + \sum_{i=1}^{N-1} T'_i + \frac{T'_N}{2}\right) - \left(\frac{T_0}{2} + \sum_{i=1}^{N-1} T_i + \frac{T_N}{2}\right)\right\}. \tag{21}$$

The initial matrix temperatures, at $\eta = 0$, are easily obtained from

$$T_0 = K_0 + t_{\text{in}} \tag{22}$$

$$T_n = (-1)^n K_0 + K_n + t_{\text{in}}. \tag{23}$$

4. NEW ALGORITHMS FOR THE UNSYMMETRIC CASE

4.1. *Method* 1

The system of equations (13) can be re-arranged to give

$$\begin{bmatrix} \mathbf{L} & \mathbf{F}' \\ \mathbf{0} & (\mathbf{L} - [\mathbf{FL}^{-1}]\mathbf{F}') \end{bmatrix}\begin{bmatrix} \underline{k} \\ \underline{k}' \end{bmatrix} = \begin{bmatrix} \underline{y} \\ -(\mathbf{I} + [\mathbf{FL}^{-1}])\underline{y} \end{bmatrix} \tag{24}$$

where $\mathbf{I}$ is the $(N+1) \times (N+1)$ identity matrix and

$$\mathbf{L}^{-1}(i,j) = \begin{cases} 1 & \text{if } i = j \\ (-1)^{i+1} & \text{if } j = 0, i \neq j \\ 0 & \text{otherwise.} \end{cases} \tag{25}$$

From equation (24) we obtain

$$(\mathbf{L} - [\mathbf{FL}^{-1}]\mathbf{F}')\underline{k}' = -(\mathbf{I} + [\mathbf{FL}^{-1}])\underline{y}. \tag{26}$$

This is a system of only $(N+1)$ equations in the $(N+1)$ unknowns K'_j $(0 \leqslant j \leqslant N)$. Equation (26) requires the product matrix $[\mathbf{FL}^{-1}]\mathbf{F}'$ to be evaluated. $[\mathbf{FL}^{-1}]$ and $\mathbf{F}'$ are both triangular matrices of the form shown in Fig. 1. An efficient multiplication algorithm can thus be derived to obtain the necessary product matrix. Once $\underline{k}'$ has been obtained $\underline{k}$ can be computed from

$$\mathbf{L}\underline{k} = \underline{y} - \mathbf{F}'\underline{k}'. \tag{27}$$

This algorithm for solving the system of equations (13) should always be employed in preference to solving the system of equations directly, as the major work load involved in solving the equations using this algorithm is

$$\frac{N^3}{3}\{\text{to obtain the product matrix}\} + \frac{N^3}{3}\{\text{to solve the resulting system of equations}\} = \frac{2N^3}{3} \tag{28}$$

while solving the $2(N+1)$ equations directly requires a work load which is proportional to $(2N)^3/3 = 8N^3/3$.

4.2. *Method* 2

A slightly different algorithm can be derived by transforming the system of equations (13) in the following manner:

$$\begin{bmatrix} \mathbf{0} & (\mathbf{F}' - [\mathbf{LF}^{-1}]\mathbf{L}) \\ \mathbf{F} & \mathbf{L} \end{bmatrix}\begin{bmatrix} \underline{k} \\ \underline{k}' \end{bmatrix} = \begin{bmatrix} (\mathbf{I} + [\mathbf{LF}^{-1}])\underline{y} \\ -\underline{y} \end{bmatrix}. \tag{29}$$

This requires the inverse matrix $\mathbf{F}^{-1}$ to be evaluated. The particular form of $\mathbf{F}$ allows $\mathbf{F}^{-1}$ to be determined algebraically

$$\mathbf{F}^{-1}(i,j) = \begin{cases} -\exp(\Pi) & \text{if } j = N, i = 0 \\ -\exp((1-a)\Pi)\zeta_i & \text{if } j = N, i > 0 \\ -\exp((1-a)\Pi) & \text{if } j < N, (i+j) = N \\ -\exp((1-a)\Pi)\beta_{i+j-N} & \text{if } j < N, (i+j) > N \\ 0 & \text{otherwise} \end{cases} \tag{30}$$

where

$$\zeta_i = (-1)^{i-1}\left\{1 + \sum_{j=1}^{i-1}(-1)^j\beta_j\right\} = -\zeta_{i-1} + \beta_{i-1} \tag{31}$$

and

$$\beta_n = -\alpha_n - \sum_{i=1}^{n-1} \alpha_{n-i}\beta_i \quad (1 \leqslant n \leqslant N-1). \quad (32)$$

From equation (29) we obtain

$$(\mathbf{F}' - [\mathbf{LF}^{-1}]\mathbf{L})\underline{k}' = (\mathbf{I} + [\mathbf{LF}^{-1}])\underline{y} \quad (33)$$

which is a system of only $(N+1)$ equations in the $(N+1)$ unknowns K_j' $(0 \leqslant j \leqslant N)$. Once $\underline{k}'$ has been obtained, $\underline{k}$ is again computed using equation (27). The only product matrices which must be evaluated here are of the form $[\mathbf{LF}^{-1}]$ and $[\mathbf{LF}^{-1}]\mathbf{L}$. As $\mathbf{L}$ is a very simple matrix ($\mathbf{L}$ is the identity matrix with elements in the first column) these product matrices are easily obtained.

The major work load associated with this method of solving the system of equations (13) is now proportional to just $N^3/3$ instead of $2N^3/3$. It turns out, however, that this latter algorithm is not so generally applicable as the first algorithm described in this section. For values of Π, Π' greater than 10, equation (33) is not amenable to solution on a digital computer because matrix $(\mathbf{F}' - [\mathbf{LF}^{-1}]\mathbf{L})$ contains both very large elements which have multipliers of the form $\exp(\Pi)$ and very small elements with $\exp(-\Pi)$ multipliers. It is for this reason that this algorithm should only be employed for values of max $(\Pi, \Pi') \leqslant 10$. This algorithm has been implemented using double precision arithmetic and it has been found that when max $(\Pi, \Pi') \leqslant 10$ the algorithm can be employed with impunity. It is known that max $(\Pi, \Pi') \leqslant 10$ covers a significant number of, but not all practical applications for regenerators. Even so, this latter algorithm should prove to be most useful. Should calculations involving larger values of Π be required, then Method 1 set out here and based on equation (26) is recommended.

5. NEW ALGORITHM FOR THE SYMMETRIC CASE

It has been shown that, in the case of the symmetric regenerator, the problem reduces to one of solving equation (19) only. Matrix $(\mathbf{L} - \mathbf{F})$ can be shown to be of the form:

M	$\mathbf{F}_1$
$\mathbf{F}_2$	**I**

Matrices $\mathbf{M}$, $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{I}$ are partitions of matrix $(\mathbf{L} - \mathbf{F})$. The dimensions of these submatrices depend on N

N is odd; dimension of all matrices is

$$\left(\frac{N+1}{2}\right) \times \left(\frac{N+1}{2}\right)$$

N is even;

dimension $\mathbf{M}$ is $\left(\frac{N}{2}+1\right) \times \left(\frac{N}{2}+1\right)$

dimension $\mathbf{F}_1$ is $\left(\frac{N}{2}+1\right) \times \left(\frac{N}{2}\right)$

dimension $\mathbf{F}_2$ is $\left(\frac{N}{2}\right) \times \left(\frac{N}{2}+1\right)$

dimension $\mathbf{I}$ is $\left(\frac{N}{2}\right) \times \left(\frac{N}{2}\right)$.

In either case, N odd or even, we obtain the solution

$$\begin{bmatrix} (\mathbf{M} - \mathbf{F}_1\mathbf{F}_2) & \mathbf{0} \\ \mathbf{F}_2 & \mathbf{I} \end{bmatrix} \begin{bmatrix} \underline{k}_1 \\ \underline{k}_2 \end{bmatrix} = \begin{bmatrix} \underline{y}_1 - \mathbf{F}_1 \underline{y}_2 \\ \underline{y}_2 \end{bmatrix} \quad (34)$$

where $\underline{k}_1$, $\underline{k}_2$ and $\underline{y}_1$, $\underline{y}_2$, are the appropriate partitions of $\underline{k}$ and $\underline{y}$, respectively; their individual dimensions are determined by whether N is odd or even.

From equation (34) we obtain

$$(\mathbf{M} - \mathbf{F}_1\mathbf{F}_2)\underline{k}_1 = \underline{y}_1 - \mathbf{F}_1 \underline{y}_2 \quad (35)$$

from which $\underline{k}_1$ can be obtained. Matrices $\mathbf{F}_1$ and $\mathbf{F}_2$ are, essentially, of the same triangular form as shown in Fig. 1 and so the product matrix $\mathbf{F}_1\mathbf{F}_2$ can be computed rapidly. Once $\underline{k}_1$ has been evaluated, $\underline{k}_2$ is obtained from

$$\underline{k}_2 = \underline{y}_2 - \mathbf{F}_2 \underline{k}_1. \quad (36)$$

Thus, when N is odd the solution of only $(N+1)/2$ equations is required and when N is even only $(N/2+1)$ equations. This effectively halves the number of equations to be solved for the symmetric case. It follows that the major work load associated with this algorithm is

$$\frac{1}{3}\left(\frac{N}{2}\right)^3 \{\text{to obtain the product matrix}\}$$

$$+\frac{1}{3}\left(\frac{N}{2}\right)^3 \{\text{to solve the resulting system of equations}\}$$

$$= \frac{N^3}{12}. \quad (37)$$

This is only one-quarter of the work load required to solve the system of equations (19) directly.

6. RICHARDSON'S EXTRAPOLATION

The Richardson extrapolation technique is applicable to the method of solution described here. It is based on the knowledge that the error in η'_{reg} introduced by the use of the trapezoidal rule (equation (5)) is proportional to $\Delta\xi^2$. This is discussed in Appendix B.

The thermal effectiveness is first computed employ-

Table 1. Relative CPU times, for the original closed method of solution of ref. [1] and the present approach

Relative CPU time for $\frac{\text{New algorithms}}{\text{Hill–Willmott (1987)}}$	$N = 10$	$N = 20$	$N = 30$	$N = 40$	$N = 50$
$\frac{\text{Solution of equation (35)}}{\text{Solution of equation (19)}}$	0.54	0.39	0.31	0.30	0.29
$\frac{\text{Solution of equation (26)}}{\text{Solution of equation (13)}}$	0.40	0.32	0.29	0.28	0.27
$\frac{\text{Solution of equation (33)}}{\text{Solution of equation (13)}}$	0.30	0.22	0.19	0.18	0.17

ing a particular steplength $\Delta\xi = \Lambda/N$ (denoted by $\eta'_{\text{reg},N}$). The value of N is then doubled, thus halving the steplength $\Delta\xi = \Lambda/2N$, and the thermal effectiveness, $\eta'_{\text{reg},2N}$, recomputed employing this smaller steplength. A value of the thermal effectiveness can be obtained, η'^{*}_{reg}, by using

$$\eta'^{*}_{\text{reg}} = \frac{4\eta'_{\text{reg},2N} - \eta'_{\text{reg},N}}{3}. \qquad (38)$$

This is significantly closer to the true value of the thermal effectiveness, η'_{reg}. Values of $N = 4$ and 8 have been found to generate accurate values of η'^{*}_{reg} over a wide range of descriptive parameters at minimal computational cost. Numerical experiments using this method are now described.

7. RESULTS

The computational speed of the original method of solution considered in ref. [1] has been compared with the method of solution considered here which employs the new matrix solving algorithms set out in Sections 4 and 5. The total CPU time required to evaluate the following cases have been computed for each of the various algorithms:

$$\left.\begin{array}{ll}\Lambda = 1,2,\ldots,10; & \Pi = 1,2,3 \\ \Lambda' = k\Lambda; & \Pi' = k\Pi\end{array}\right\}$$

$$\begin{array}{ll} k = 1 & \text{for the symmetric case} \\ k = 2 & \text{for the unsymmetric case.}\end{array} \qquad (39)$$

In this manner the relative speeds of the various algorithms have been evaluated and these are shown in Table 1. It is clear from Table 1 that the implementation of the new matrix solving algorithms described in Sections 4 and 5 results in a considerable reduction in the computing requirements of the closed method of solution. For $N \geqslant 20$ a reduction in the CPU requirements of between 61 and 83% is observed.

The computational speed of the improved method of solution considered here has also been compared with the open method of ref. [7]. In that scheme both equations (1) and (2) are replaced employing the trapezoidal representation and thus the error associated with the finite difference form of equation (2) is the same for both the open scheme and the method of solution considered here. For the open method M distance steps and P time steps are employed to construct a time–space mesh. As the error associated with the representation of equation (2) is the same for both schemes, the value of M employed here for the open method is set equal to the value of N employed in equation (5). In the open method the number of time steps is also set equal to the number of distance steps.

The CPU time required to evaluate cases (39) for both the open and closed schemes have been computed. The relative speeds of these two methods are shown in Table 2. It is clear from Table 2 that the closed method of solution incorporating the algorithms set out in Sections 4 and 5 enables the cyclic equilibrium performance of the regenerators defined by cases (39) to be computed more rapidly than the open scheme. Indeed, for small values of N ($N = 10$) the closed method of solution proves to be more than an order of magnitude faster than the open scheme.

The incorporation of the algorithms described in Sections 4 and 5 enables considerable reductions to be made in the computational requirements of the closed method of solution described here. The application of the Richardson extrapolation technique improves matters still further. In Section 6 it was shown how extrapolated values of the thermal effectiveness, η'^{*}_{reg}, can be obtained employing equation (38). Equation (38) enables accurate estimates of η'_{reg}

Table 2. Relative CPU times, for the open scheme of ref. [7] and the present approach

$\frac{\Lambda'}{\Lambda} = \frac{\Pi'}{\Pi} = k$	$\frac{\text{CPU time Hill–Willmott (1988)}}{\text{CPU time Willmott (1964)}}$ $N = 10$	$N = 20$	$N = 30$
1	0.041	0.068	0.11
2	0.09	0.18	0.31

to be obtained whilst employing only low values of N. For practical calculations it has been found that employing values of $N = 4$ and 8 enables accurate estimates of η_{reg} to be obtained at a small computational cost.

In order to exhibit the speed and accuracy of this technique for evaluating η'_{reg} a comparison has been made with the approach considered by Baclic [8]. Baclic suggested the following expression for computing η'_{reg} over a wide range of dimensionless parameters $\Lambda = \Lambda'$ and $\Pi = \Pi'$:

$$\eta'_{reg} = \frac{\Lambda}{\Pi}\,\frac{1+7\beta_2-24[B-2\{R_1-A_1-90(N_1+2E)\}]}{1+9\beta_2-24[B-6\{R-A-20(N-3E)\}]}. \quad (40)$$

The definition and evaluation of the variables used in this expression are given in Appendix C.

In order to assess the accuracy and computational speed of the present approach, η'_{reg} has been computed employing equation (38) with $N = 4$ and 8 for the following range of dimensionless parameters:

$$(\Lambda = \Lambda') \leqslant 500$$
$$(\Pi = \Pi') \leqslant 50$$
$$0 \leqslant \Pi/\Lambda \leqslant 1. \quad (41)$$

The extrapolated η'_{reg} values computed using the Richardson technique are presented in Table 3. The errors in these η'_{reg} values, given by η'_{reg} (exact)† $-\eta'_{reg}$ (Table 3), are also presented in Table 4. Blank areas in Table 4 indicate errors in η'_{reg} of less than 0.0001 in magnitude while the boxed area highlights errors in η'_{reg} which are greater than 0.001 in magnitude. It is clear from Table 4 that the extrapolation procedure enables accurate estimates of η'_{reg} to be obtained over a large range of the descriptive parameters, the error in η'_{reg} being, in general, less than 0.001 in magnitude.

In a similar manner the errors in the η'_{reg} values computed employing Baclic's equation (40) have been evaluated and these are presented in Table 5. A comparison of Tables 4 and 5 indicates clearly that the extrapolation technique described here enables η'_{reg} to be computed with a greater degree of accuracy than that afforded by equation (40). This improved accuracy is achieved at no extra computational cost. The CPU time required to evaluate η'_{reg} employing equation (40) is approximately the same‡ as that required employing the extrapolation procedure (for the symmetric case) with $N = 4$ and 8.

The significance of the accuracy of the approach adopted here for providing estimates of η'_{reg} can be exhibited by considering the following definition of η'_{reg}:

$$\eta'_{reg} = \frac{\bar{t}'_{out} - t'_{in}}{t_{in} - t'_{in}} \quad (42)$$

where $\bar{t}'_{out}$ is the time average exit fluid temperature during the cold period. For the case $\Lambda = 30$, $\Pi = 12$ with $t_{in} = 1250°C$ and $t'_{in} = 10°C$, the η'_{reg} value computed employing equation (40) corresponds to an error of 38°C in the value of $\bar{t}'_{out}$ while the η'_{reg} value computed employing equation (38) with $N = 4$ and 8 corresponds to an error of only 1°C in the value of $\bar{t}'_{out}$.

The new algorithm described in Section 5 for the solution of equation (19) coupled with the Richardson extrapolation procedure enables very accurate estimates of η'_{reg} to be computed in a rapid fashion and for a large range of the descriptive parameters. In order to extend the technique to the unsymmetric case two approaches are possible.

First, the approach suggested by Hausen [9] and extended by Razelos [3] can be adopted. In this approach the unsymmetric regenerator is approximated using an 'equivalent' symmetric regenerator. The descriptive parameters for the symmetric regenerator are the harmonic means of Π, Π', Λ and Λ' defined by

$$\Pi_H = \frac{2}{\dfrac{1}{\Pi} + \dfrac{1}{\Pi'}} \quad (43)$$

$$\Lambda_H = \frac{2\Pi_H}{\dfrac{\Pi}{\Lambda} + \dfrac{\Pi'}{\Lambda'}}. \quad (44)$$

The value of η'_{reg} for the unsymmetric regenerator is then approximated using the value of η'_{reg} computed for the symmetric regenerator (denoted by $\eta'_{reg,H}$) in the following manner:

$$\eta'_{reg} = \gamma\eta_{reg} \approx \gamma\frac{1-\exp[G(1-\gamma^2)/2\gamma]}{\gamma-\exp[G(1-\gamma^2)/2\gamma]} \quad (45)$$

where

$$G = \frac{\eta'_{reg,H}}{(1-\eta'_{reg,H})}, \quad \gamma = \frac{\Pi}{\Lambda}\frac{\Lambda'}{\Pi'}. \quad (46)$$

Razelos investigated the validity of this approach and found that, in general, η'_{reg} could be computed to within 2% of its correct value using equation (45). We have confirmed Razelos' important observations in this matter. It should be noted, however, that the values of $\eta'_{reg,H}$ supplied to equation (45) through equations (46) are assumed to be sufficiently accurate for this overall error in the estimated value of η'_{reg} to be realized.

Second, it is possible using the techniques described here to retain the general, unsymmetric model without significant additional computational cost. While it is clearly very useful to be able to use the Λ_H, Π_H sym-

† The 'exact' values of η'_{reg} have been computed here employing equation (38) with $N = 40$ and 80. These exact values of η'_{reg} have been confirmed (to four decimal places) using the closed method of Iliffe [6] with 80 points in the quadrature scheme employed by that author.

‡ Both methods required approximately 5 CPU seconds to evaluate the 439 η'_{reg} values in Table 3 on a Sun 3/50 work station.

Table 3. Values of η'^{*}_{reg} for regenerators computed using equation (38) with $N = 4$ and 8

					$\frac{\Pi}{\Lambda}$ Ratio						
Λ	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
1.0	1/3	0.3332	0.3329	0.3323	0.3315	0.3304	0.3292	0.3277	0.3260	0.3241	0.3221
1.5	3/7	0.4283	0.4276	0.4264	0.4248	0.4227	0.4202	0.4173	0,4139	0.4102	0.4061
2.0	1/2	0.4996	0.4986	0.4986	0.4943	0.4912	0.4874	0.4830	0.4780	0.4725	0.4665
2.5	5/9	0.5551	0.5537	0.5513	0.5481	0.5440	0.5391	0.5333	0.5269	0.5197	0.5120
3.0	3/5	0.5994	0.5977	0.5949	0.5910	0.5861	0.5802	0.5733	0.5656	0.5570	0.5477
3.5	7/11	0.6357	0.6338	0.6305	0.6262	0.6204	0.6137	0.6058	0.5970	0.5873	0.5766
4.0	2/3	0.6659	0.6638	0.6602	0.6553	0.6491	0.6416	0.6330	0.6232	0.6124	0.6007
4.5	9/13	0.6915	0.6892	0.6853	0.6800	0.6733	0.6652	0.6559	0.6454	0.6337	0.6210
5.0	5/7	0.7134	0.7109	0.7068	0.7012	0.6941	0.6856	0.6757	0.6645	0.6521	0.6385
5.5	11/15	0.7324	0.7298	0.7255	0.7196	0.7122	0.7033	0.6929	0.6812	0.6681	0.6538
6.0	3/4	0.7491	0.7463	0.7418	0.7357	0.7280	0.7188	0.7081	0.6959	0.6823	0.6673
6.5	13/17	0.7637	0.7609	0.7562	0.7499	0.7421	0.7326	0.7216	0.7090	0.6949	0.6793
7.0	7/9	0.7768	0.7738	0.7691	0.7626	0.7546	0.7449	0.7337	0.7208	0.7062	0.6900
7.5	15/19	0.7884	0.7854	0.7806	0.7740	0.7658	0.7560	0.7446	0.7314	0.7165	0.6998
8.0	4/5	0.7989	0.7909	0.7909	0.7843	0.7760	0.7661	0.7545	0.7411	0.7258	0.7087
8.5	17/21	0.8084	0.8053	0.8003	0.7936	0.7852	0.7753	0.7635	0.7499	0.7343	0.7168
9.0	9/11	0.8171	0.8139	0.8088	0.8021	0.7937	0.7836	0.7718	0.7580	0.7422	0.7243
9.5	19/23	0.8250	0.8217	0.8166	0.8098	0.8014	0.7913	0.7794	0.7655	0.7494	0.7312
10.0	5/6	0.8322	0.8289	0.8238	0.8170	0.8086	0.7984	0.7865	0.7724	0.7562	0.7376
10.5	21/25	0.8388	0.8355	0.8304	0.8236	0.8151	0.8050	0.7930	0.7789	0.7624	0.7436
11.0	11/13	0.8450	0.8416	0.8365	0.8297	0.8212	0.8111	0.7991	0.7849	0.7683	0.7492
11.5	23/27	0.8507	0.8473	0.8421	0.8353	0.8269	0.8168	0.8048	0.7905	0.7738	0.7545
12.0	6/7	0.8559	0.8526	0.8474	0.8406	0.8322	0.8221	0.8101	0.7958	0.7789	0.7594
12.5	25/29	0.8609	0.8575	0.8523	0.8455	0.8372	0.8271	0.8151	0.8008	0.7838	0.7640
13.0	13/15	0.8654	0.8620	0.8569	0.8501	0.8418	0.8318	0.8198	0.8054	0.7884	0.7684
13.5	27/31	0.8697	0.8663	0.8612	0.8544	0.8462	0.8362	0.8242	0.8099	0.7927	0.7725
14.0	7/8	0.8738	0.8703	0.8652	0.8585	0.8503	0.8404	0.8284	0.8141	0.7968	0.7765
14.5	29/33	0.8775	0.8741	0.8690	0.8623	0.8542	0.8443	0.8324	0.8180	0.8007	0.7802
15.0	15/17	0.8811	0.8777	0.8726	0.8660	0.8578	0.8480	0.8362	0.8218	0.8044	0.7837
15.5	31/35	0.8845	0.8810	0.8759	0.8694	0.8613	0.8515	0.8397	0.8254	0.8080	0.7871
16.0	8/9	0.8876	0.8842	0.8791	0.8726	0.8646	0.8549	0.8431	0.8288	0.8114	0.7904
17.0	17/19	0.8935	0.8901	0.8850	0.8786	0.8707	0.8611	0.8495	0.8352	0.8177	0.7964
18.0	9/10	0.8987	0.8953	0.8904	0.8840	0.8762	0.8668	0.8553	0.8411	0.8235	0.8020
19.0	19/21	0.9035	0.9001	0.8952	0.8889	0.8812	0.8719	0.8606	0.8464	0.8288	0.8071
20	10/11	0.9078	0.9045	0.8996	0.8934	0.8858	0.8767	0.8654	0.8514	0.8338	0.8118
25	25/27	0.9246	0.9214	0.9169	0.9111	0.9041	0.8956	0.8851	0.8716	0.8540	0.8312
30	15/16	0.9362	0.9332	0.9290	0.9236	0.9170	0.9091	0.8994	0.8865	0.8690	0.8455
35	35/37	0.9447	0.9419	0.9380	0.9328	0.9266	0.9194	0.9103	0.8981	0.8807	0.8565
40	20/21	0.9511	0.9485	0.9448	0.9400	0.9342	0.9274	0.9190	0.9073	0.8901	0.8654
45	45/47	0.9562	0.9537	0.9503	0.9457	0.9402	0.9339	0.9261	0.9149	0.8979	0.8727
50	25/26	0.9603	0.9580	0.9548	0.9504	0.9452	0.9393	0.9320	0.9213	0.9044	0.8787
60	30/31	0.9666	0.9645	0.9616	0.9576	0.9529	0.9477	0.9413	0.9315		
70	35/36	0.9711	0.9693	0.9666	0.9630	0.9586	0.9540	0.9484			
80	40/41	0.9746	0.9729	0.9705	0.9671	0.9630	0.9590				
90	45/46	0.9773	0.9757	0.9735	0.9703	0.9666					
100	50/51	0.9795	0.9780	0.9759	0.9729	0.9694					
150	75/76	0.9861	0.9850	0.9835							
200	100/101	0.9895	0.9886								
300	150/151	0.9930									
400	200/201	0.9947									
500	250/251	0.9958									

metric approximation to the general non-symmetric case in order to obtain η'_{reg} and η_{reg} using equation (45), it is still necessary, however, to compute the solutions to equations (1) and (2) for the non-symmetric case directly if the spatial and chronological variations of gas and solid temperature at cyclic equilibrium are required.

In this context of the non-symmetric case, the advantages of the Richardson extrapolation method are clearly apparent. The thermal effectiveness $\eta'_{reg,N}$ and $\eta'_{reg,2N}$ are now computed according to equation (38) employing the algorithms set out in Section 4. Again, only low values of N are required to compute accurate estimates of the thermal effectiveness η'_{reg}.

Table 4. Values of η'_{reg} (exact) $-\ \eta'_{reg}$ (Table 3)

Λ	$\frac{\Pi}{\Lambda}$ Ratio 0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
1.0											
1.5											
2.0											
2.5											
3.0											
3.5											
4.0											
4.5											
5.0											
5.5											
6.0											
6.5											
7.0											
7.5											
8.0											
8.5											
9.0											
9.5											
10.0											
10.5										−0.0001	−0.0001
11.0										−0.0001	−0.0001
11.5										−0.0001	−0.0001
12.0									−0.0001	−0.0001	−0.0002
12.5									−0.0001	−0.0002	−0.0002
13.0									−0.0001	−0.0002	−0.0002
13.5									−0.0001	−0.0002	−0.0002
14.0									−0.0002	−0.0002	−0.0002
14.5									−0.0002	−0.0002	−0.0002
15.0									−0.0002	−0.0003	−0.0002
15.5									−0.0002	−0.0003	−0.0003
16.0								−0.0001	−0.0002	−0.0003	−0.0003
17.0								−0.0001	−0.0003	−0.0004	−0.0003
18.0								−0.0001	−0.0003	−0.0004	−0.0003
19.0						0.0001		−0.0002	−0.0004	−0.0005	−0.0004
20						0.0001		−0.0002	−0.0004	−0.0005	−0.0004
25					0.0001	0.0002		−0.0003	−0.0007	−0.0008	−0.0005
30					0.0001	0.0002		−0.0005	−0.0010	−0.0010	−0.0004
35					0.0002	0.0003		−0.0007	−0.0012	−0.0012	−0.0003
40					0.0002	0.0003		−0.0009	−0.0015	−0.0013	−0.0001
45					0.0002	0.0004		−0.0011	−0.0018	−0.0014	−0.0002
50					0.0002	0.0004	−0.0001	−0.0013	−0.0020	−0.0015	0.0005
60					0.0003	0.0005	−0.0002	−0.0016	−0.0024		
70					0.0003	0.0005	−0.0003	−0.0019			
80					0.0003	0.0005	−0.0004				
90					0.0003	0.0005					
100					0.0003	0.0005					
150											
200											
300											
400											
500											

Moreover, the extrapolation procedure can then be applied also to the solid and fluid temperatures in order to yield a complete space–time temperature profile of the regenerator. In high temperature regenerator applications it may be necessary to provide estimates of the time variations in the solid and fluid temperatures at the entrance and exit of the regenerator or the variation of, say, the solid temperature down the length of the regenerator at some instance in time, for example in the middle of the hot and cold periods.

Table 6 shows how the extrapolation procedure can be applied to the spatial variation of solid and fluid temperatures for an unsymmetric regenerator. In this manner the solid and fluid temperatures at any position along the length of the regenerator and at any

Table 5. Values of η'_{reg} (exact) $-\eta'_{reg}$ (equation (46))

	$\frac{\Pi}{\Lambda}$ Ratio										
Λ	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
1.0											
1.5											
2.0											
2.5											
3.0											
3.5											
4.0											
4.5											
5.0											
5.5											
6.0						0.0001	0.0001	0.0001			
6.5					0.0001	0.0002	0.0002	0.0001	0.0001		
7.0					0.0002	0.0002	0.0002	0.0002	0.0001		
7.5				0.0001	0.0002	0.0003	0.0002	0.0002	0.0001		
8.0				0.0002	0.0003	0.0003	0.0003	0.0002	0.0001		
8.5				0.0002	0.0003	0.0004	0.0003	0.0002	0.0001		
9.0				0.0002	0.0004	0.0004	0.0004	0.0002	0.0001		
9.5				0.0003	0.0004	0.0005	0.0004	0.0003	0.0001		
10.0			0.0001	0.0003	0.0005	0.0005	0.0004	0.0003	0.0001		
10.5			0.0001	0.0004	0.0006	0.0006	0.0005	0.0003	0.0001		
11.0			0.0001	0.0004	0.0006	0.0006	0.0005	0.0003	0.0001		
11.5			0.0002	0.0005	0.0007	0.0007	0.0005	0.0003	0.0001		
12.0			0.0002	0.0005	0.0008	0.0008	0.0006	0.0003	0.0001		
12.5			0.0002	0.0006	0.0009	0.0008	0.0006	0.0003	0.0001		
13.0			0.0002	0.0006	0.0009	0.0009	0.0006	0.0003	0.0001		
13.5			0.0003	0.0007	0.0010	0.0009	0.0006	0.0003	0.0001		
14.0			0.0003	0.0008	0.0011	0.0010	0.0007	0.0003	0.0001		
14.5			0.0003	0.0008	0.0011	0.0010	0.0007	0.0003	0.0001		
15.0			0.0003	0.0009	0.0012	0.0011	0.0007	0.0003	0.0001		
15.5			0.0004	0.0010	0.0013	0.0011	0.0007	0.0003	0.0001		0.0001
16.0			0.0004	0.0010	0.0014	0.0012	0.0007	0.0003	0.0001		0.0001
17.0			0.0005	0.0012	0.0015	0.0013	0.0008	0.0003	0.0001		0.0002
18.0			0.0005	0.0013	0.0017	0.0014	0.0008	0.0003	0.0001	0.0001	0.0002
19.0			0.0006	0.0014	0.0018	0.0015	0.0008	0.0003	0.0001	0.0002	0.0002
20			0.0006	0.0016	0.0019	0.0015	0.0008	0.0003	0.0002	0.0002	0.0003
25		0.0001	0.0010	0.0022	0.0026	0.0019	0.0009	0.0004	0.0003	0.0004	0.0005
30		0.0002	0.0012	0.0027	0.0031	0.0021	0.0010	0.0005	0.0005	0.0007	0.0008
35		0.0002	0.0015	0.0032	0.0035	0.0024	0.0011	0.0006	0.0007	0.0010	0.0011
40		0.0003	0.0018	0.0037	0.0039	0.0026	0.0012	0.0007	0.0010	0.0013	0.0013
45		0.0003	0.0020	0.0041	0.0043	0.0027	0.0013	0.0009	0.0012	0.0017	0.0015
50		0.0003	0.0022	0.0044	0.0045	0.0029	0.0014	0.0010	0.0015	0.0020	0.0018
60		0.0004	0.0026	0.0050	0.0050	0.0031	0.0016	0.0014	0.0021		
70		0.0005	0.0029	0.0055	0.0054	0.0033	0.0018	0.0018			
80		0.0006	0.0031	0.0059	0.0058	0.0035	0.0020				
90		0.0006	0.0033	0.0062	0.0060	0.0036					
100		0.0007	0.0035	0.0065	0.0063	0.0038					
150		0.0009	0.0042	0.0075							
200		0.0010	0.0045								
300		0.0012									
400		0.0013									
500		0.0013									

time during the period can be computed in an accurate and efficient manner.

8. CONCLUSIONS

It is important to have some overall view of the significance of this work. Much has been published over the last 60 years on the solution of equations (1) and (2) and it is perhaps worthwhile setting the current work in context. Apart from the so-called 'open' methods, in which the regenerator model is cycled to equilibrium, all the methods, like the one described here, are 'closed' in that the cyclic equilibrium performance of the regenerator is computed directly as a boundary value problem. The majority of methods including those of Iliffe [6] and Baclic [8] are based on

Table 6. The Richardson extrapolation technique applied to the solid and fluid temperatures during the cold period for the unsymmetric regenerator $\Lambda = 12$, $\Lambda' = 10$, $\Pi = 3$, $\Pi' = 4$

	Gas				Solid			
ξ/Λ	$N = 4$	$N = 8$	Extrap.	$N = 48$	$N = 4$	$N = 8$	Extrap.	$N = 48$
				$\tau' = 0$				
0.00	0.00000	0.00000	0.0000	0.0000	0.04520	0.04903	0.0503	0.0502
0.25	0.09767	0.09824	0.0984	0.0987	0.13061	0.13887	0.1416	0.1409
0.50	0.23794	0.24115	0.2422	0.2422	0.31722	0.31780	0.3180	0.3187
0.75	0.51091	0.49915	0.4952	0.4965	0.65000	0.62958	0.6228	0.6243
1.00	0.84595	0.84531	0.8451	0.8451	0.97489	0.97478	0.9747	0.9747
				$\tau' = \Pi'/2$				
0.00	0.00000	0.00000	0.0000	0.0000	0.00612	0.00664	0.0068	0.0068
0.25	0.04016	0.04300	0.0440	0.0437	0.06616	0.07008	0.0714	0.0710
0.50	0.13748	0.13940	0.1400	0.1400	0.18933	0.19106	0.1916	0.1918
0.75	0.32183	0.31667	0.3150	0.3157	0.41746	0.40872	0.4058	0.4068
1.00	0.59479	0.59305	0.5925	0.5925	0.71753	0.71635	0.7160	0.7160
				$\tau' = \Pi'$				
0.00	0.00000	0.00000	0.0000	0.0000	0.00083	0.00090	0.0009	0.0009
0.25	0.01651	0.01674	0.0168	0.0168	0.02889	0.03039	0.0308	0.0307
0.50	0.07282	0.07394	0.0743	0.0743	0.10550	0.10706	0.1076	0.1075
0.75	0.19437	0.19209	0.1913	0.1916	0.25893	0.25509	0.2538	0.2543
1.00	0.39763	0.39512	0.3943	0.3943	0.49567	0.49351	0.4928	0.4928

integral equation formulations of equations (1) and (2).

The integral equation approach has three basic weaknesses. Firstly, the methods including that of Iliffe [6] break down for large values of the ratio Λ/Π. This has been discussed in ref. [10]. This is not as important, however, as, secondly, the requirement to double the number of simultaneous linear equations which must be solved to tackle the general problem as opposed to the symmetric case ($\Lambda = \Lambda'$, $\Pi = \Pi'$). The significance of the work described here is that both of these problems have been overcome without resort to the use of Λ_H and Π_H (see equations (43) and (44)).

Finally, the third, most important problem is that the published literature reveals that no one has been able to incorporate, explicitly within the integral equation formulation (or any other closed method) of the regenerator model, any nonlinearities including, for example, the temperature dependence of the thermophysical properties of the gases and the packing. Kulakowski and Anielewski [11] applied mean values of such properties to the Iliffe scheme, which were corrected iteratively to allow some presentation of their temperature dependence. The linear form of the Iliffe method remained intact however.

The approach of ref. [1], from which the present work follows, can be developed to permit such nonlinearities to be incorporated explicitly, however, within differential equations (7) and (8). This will be discussed in another paper.

From a practical point of view, the strength of the approach described here is the speed and accuracy of the calculations and the robustness of the method. Two points must be conceded, however.

(1) The linear model allows only an approximate representation of the real world and attention must be given to the incorporation of nonlinearities in the model, as discussed above. Even so, the linear model can be used to yield first approximations of either designs of new regenerators or the performance of existing regenerators. Such initial approximations can be refined, if necessary, using a more complex nonlinear model.

(2) For single calculations of regenerator performance, using the linear model, for typical ranges of the dimensionless parameters, existing methods such as those of Baclic [8] and Iliffe [6] are more than adequate.

In CAD (computer aided design) software, however, optimal regenerator designs relative to user specified criteria and constraints, can be sought using search techniques. Here it is necessary to perform tens or even hundreds of calculations of the performance of different regenerator arrangements, to find a single optimal design. The designer may well wish to explore the effect of modifying the design criteria and the design constraints, in which case these search techniques must be applied several times. It is in these circumstances that the computational advantages of the methods described here can be exploited.

It turns out that the linear model is significantly computationally cheaper to use than the non-linear model, as one might expect. As a consequence, it is likely that CAD software will incorporate both the rapid methods of calculation described here to yield initial designs which can be refined, if necessary, employing more realistic models.

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APPENDIX A. ALGORITHM FOR THE α_r VALUES

$$\alpha_r = b^r \sum_{k=1}^{r} \binom{r-1}{k-1} \left(\frac{a(1+b)\eta}{b}\right)^k \frac{1}{k!} \quad (1 \leqslant r \leqslant N-1).$$

Defining

$$\theta_{r,j} = b^r \sum_{k=j}^{r} \binom{r-j}{k-j} \left(\frac{a(1+b)\eta}{b}\right)^k \frac{1}{k!}$$

then

$$\alpha_r = \theta_{r,1}. \tag{A1}$$

Applying the binomial relationship

$$\binom{r}{k} = \binom{r-1}{k} + \binom{r-1}{k-1}$$

to the expression for $\theta_{r,j}$ yields

$$\theta_{r,j} = b^r \sum_{k=j}^{r} \binom{r-j-1}{k-j} \left(\frac{a(1+b)\eta}{b}\right)^k \frac{1}{k!} + b^r \sum_{k=j}^{r} \binom{r-j-1}{k-j-1} \left(\frac{a(1+b)\eta}{b}\right)^k \frac{1}{k!}.$$

Note that the first series in this expression only goes as far as $(r-1)$ and that the second series only starts at $(k = j+1)$. This expression can be rewritten as

$$\theta_{r,j} = b\left\{b^{(r-1)} \sum_{k=j}^{(r-1)} \binom{(r-1)-j}{k-j} \left(\frac{a(1+b)\eta}{b}\right)^k \frac{1}{k!}\right\} + b^r \sum_{k=(j+1)}^{r} \binom{r-(j+1)}{k-(j+1)} \left(\frac{a(1+b)\eta}{b}\right)^k \frac{1}{k!}$$

which is simply

$$\theta_{r,j} = b\theta_{r-1,j} + \theta_{r,j+1}. \tag{A2}$$

We also have

$$\alpha_1 = \theta_{1,1} = a(1+b)\eta$$

$$\theta_{r,r} = \frac{[a(1+b)\eta]^r}{r!} = \left(\frac{a(1+b)\eta}{r}\right)\theta_{r-1,r-1}. \tag{A3}$$

Equations (A1)–(A3) enable the α_r values to be evaluated in an efficient manner.

APPENDIX B. RICHARDSON'S EXTRAPOLATION

When the trapezoidal rule is employed to obtain a solution, $y(x)$, to the initial value problem given by

$$\frac{dy}{dx} = f(x, y) \quad \text{and } y(0) = y_0$$

it can be shown to yield an asymptotic expression for the approximate value of $y(x)$ computed using a steplength h (denoted by $y(x; h)$) of the form

$$y(x;h) \approx y(x) + A_1h^2 + A_2h^4 + \cdots$$

(see pp. 190–191 of ref. [4]). As $h \to 0$, $y(x;h) \to y(x)$ as required. Now if both $y(x;h)$ and $y(x;h/2)$ are computed then

$$y(x;h) \approx y(x) + A_1h^2 + A_2h^4 + \cdots$$

$$y(x;h/2) \approx y(x) + A_1\frac{h^2}{4} + A_2\frac{h^4}{16} + \cdots.$$

Multiplying the last equation by 4 and subtracting the previous equation from the resulting expression yields

$$4y(x;h/2) - y(x;h) \approx 3y(x) + O(h^4)$$

where $O(h^4)$ indicates terms of the order of h^4. Thus, if we employ

$$y^* = \frac{4y(x;h/2) - y(x;h)}{3}$$

a much closer approximation, y^*, to $y(x)$ than either $y(x;h)$ or $y(x;h/2)$ is obtained. The associated errors are of $O(h^4)$ for y^* and of $O(h^2)$ for both $y(x;h)$ and $y(x;h/2)$. This is the *deferred approach to the limit*, alternatively named *Richardson's extrapolation.*

APPENDIX C

Baclic [8] suggested the following expression for the thermal effectiveness:

$$\eta'_{\text{reg}} = \frac{\Lambda}{\Pi} \frac{1 + 7\beta_2 - 24[B - 2\{R_1 - A_1 - 90(N_1 + 2E)\}]}{1 + 9\beta_2 - 24[B - 6\{R - A - 20(N - 3E)\}]}$$

where

$$B = 3\beta_3 - 13\beta_4 + 30(\beta_5 - \beta_6)$$
$$R = \beta_2[3\beta_4 - 5(3\beta_5 - 4\beta_6)]$$
$$A = \beta_3[3\beta_3 - 5(3\beta_4 + 4\beta_5 - 12\beta_6)]$$
$$N = \beta_4[2\beta_4 - 3(\beta_5 + \beta_6)] + 3\beta_5^2$$
$$E = \beta_2\beta_4\beta_6 - \beta_2\beta_5^2 - \beta_3^2\beta_6 + 2\beta_3\beta_4\beta_5 - \beta_4^3$$
$$N_1 = \beta_4[\beta_4 - 2(\beta_5 + \beta_6)] + 2\beta_5^2$$
$$A_1 = \beta_3[\beta_3 - 15(\beta_4 + 4\beta_5 - 12\beta_6)]$$
$$R_1 = \beta_2[\beta_4 - 15(\beta_5 - 2\beta_6)]$$
$$\beta_m = V_m(\Pi, \Lambda)/\Lambda^{m-1}, m = 2, 3, \ldots, 6.$$

V_m is a special function given by

$$V_m(x, y) = e^{-(x+y)} \sum_{n=m-1}^{\infty} \binom{n}{m-1} \left(\frac{y}{x}\right)^{n/2} I_n(2\sqrt{(xy)}) \quad (m \geqslant 1)$$

and $I_n(\ )$ is the modified Bessel function of nth order.

Employing the binomial relationship

$$\binom{n}{m-1} = \binom{n-1}{m-1} + \binom{n-1}{m-2}$$

it can be shown that

$$V_m(x, y) = G_{m-1}(x, y) + G_{m-2}(x, y) \quad (m \geq 1)$$

in which $G_m(x, y)$ is given by

$$G_m(x, y) = e^{-(x+y)} \sum_{n=m}^{\infty} \binom{n}{m} \left(\frac{y}{x}\right)^{(n+1)/2} I_{n+1}(2\sqrt{(xy)}) \quad (m \geq 0).$$

In this paper the functions $G_m(x, y)$ were evaluated using the expressions presented by Romie [12]

$$G_{-1}(x, y) = e^{-(x+y)} \sum_{r=0}^{\infty} \frac{x^r}{r!} \frac{y^r}{r!}$$

$$G_{-2}(x, y) = e^{-(x+y)} \sum_{r=0}^{\infty} \frac{x^{r+1}}{(r+1)!} \frac{y^r}{r!} - G_{-1}(x, y)$$

$$G_0(x, y) = e^{-(x+y)} \sum_{r=0}^{\infty} \frac{y^{r+1}}{(r+1)!} \sum_{p=0}^{r} \frac{x^p}{p!}$$

$$G_{n+1}(x, y) = \frac{1}{n+1}[(y-x-1-2n)G_n(x, y) + (2y-n)G_{n-1}(x, y) + yG_{n-2}(x, y)] \quad (n \geq 0).$$

The recurrence relation for $G_{n+1}(x, y)$ is valid here as n is small.

CALCULS RAPIDES ET PRECIS DES REGENERATEURS THERMIQUES

Résumé—On décrit des moyens qui augmentent considérablement la vitesse de calcul des caractéristiques des régénérateurs thermiques, par l'utilisation de la méthode de Hill et Willmott (*Int. J. Heat Mass Transfer* **30,** 241–249 (1987)) laquelle a été développée à partir de la méthode de Razelos (*Wärme- und Stoffübertr.* **12,** 59–71 (1979)). En particulier le nombre d'équations linéaires simultanées à résoudre pour le cas non symétrique est divisé par deux. On tire avantage de la méthode d'extrapolation de Richardson.

GENAUE UND SCHNELLE BERECHNUNG VON THERMISCHEN REGENERATOREN

Zusammenfassung—Es werden Wege beschrieben, wodurch die Geschwindigkeit der Leistungsberechnung thermischer Regeneratoren mit der Methode von Hill und Willmott (*Int. J. Heat Mass Transfer* **30,** 241–249 (1987)) beträchtlich erhöht werden kann, welche aus der Methode von Razelos entwickelt wurde (*Wärme- und Stoffübertr.* **12,** 59–71 (1979)). Insbesondere wird die Anzahl der simultanen linearen Gleichungen, die für den nicht-symmetrischen Fall gelöst werden müssen, halbiert. Hierzu wurde Richardson's Extrapolationsmethode benutzt.

ТОЧНЫЕ И БЫСТРЫЕ РАСЧЕТЫ ТЕПЛОВОГО РЕГЕНЕРАТОРА

Аннотация—Предложены спосбы ускорения расчетов работы тепловых регенераторов, использующие метод Хилла и Виллмотта (*Int. J. Heat Mass Transfer* **30,** 241–249 (1987)), разработанный на основе метода Рацелоса (*Wärme- und Stoffübertr.* **12,** 59–71 (1979)). В частности, число совместных линейных уравнений, решаемых в несимметричном случае, сокращается вдвое. Используется метод экстраполяции Ричардсона.